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SENSITIVITY OF THE GRADIENT WIND FIELD RELATIVE TO PRESSURE VARIATIONS

BY

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PREFACE

This paper presents the results of a study of the gradient wind field in large scale atmospheric motions and its sensitivity relative to pressure variations. The objectives were: to determine the factors that affect the sensitivity; to compare qualitatively the sensitivity of tho gradient wind and the effects of these factors in certain representative pressure fields; and to compare qualitatively the sensitivity of the gradient wind with the sensitivity of the geostrophic wind as presented by S. Petterssen in a previous study.

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TABLE OF SYMBOLS AND ABBREVIATIONS

$\mathfrak{l}$	Unit vector along the x-axis
$\mathfrak{m}$	Unit vector along the y-axis
$\mathfrak{k}$	Unit vector in the vertical
$\mathbb{V}$	$(u\mathfrak{l}, v\mathfrak{m})$ Vector wind velocity
$\dot{\mathbb{V}}$	$(\dot{u}\mathfrak{l}, \dot{v}\mathfrak{m})$ Vector acceleration
$\ddot{\mathbb{V}}$	$(\ddot{u}\mathfrak{l}, \ddot{v}\mathfrak{m})$ Derivative of the vector acceleration
V	(u, v, w) Wind velocity
$\dot{V}$	$(\dot{u}, \dot{v}, \dot{w})$ Acceleration
$\ddot{V}$	$(\ddot{u}, \ddot{v}, \ddot{w})$ Derivative of the acceleration
V_g	(u_g, v_g) Geostrophic wind
V_{gr}	(u_{gr}, v_{gr}) Gradient wind
X	Horizontal component of the pressure force along the x-axis
Y	Horizontal component of the pressure force along the y-axis
K_i	Tangential curvature of the streamlines, positive for cyclonic curvature and negative for anticyclonic curvature
K_n	Orthogonal curvature of the streamlines, positive for converging streamlines and negative for diverging streamlines
K_T	Curvature of the trajectory of the air particle, positive for cyclonic curvature and negative for anticyclonic curvature
λ	$(= 2\omega \sin \phi)$ z-component of the coriolis acceleration
λ'	$(= 2\omega \cos \phi)$ y-component of the coriolis acceleration
E	Mean radius of the earth
g	Acceleration of gravity
ϕ	Angle of latitude

Ω	Angular speed of the earth's rotation
I_x	Analogous to x-component of the isallobaric wind as defined by Brunt and Douglas [1]
I_y	Analogous to y-component of the isallobaric wind as defined by Brunt and Douglas [1]
m	Factor of proportionality between the pressure force and the centripetal acceleration in gradient flow
M	$= 1 - m$
T	Temperature
c_p	Specific heat at constant pressure
Z_θ	Height above the reference level of an isentropic surface
ψ	The Montgomery acceleration potential for an isentropic surface (1937) [4]

$$\psi = c_p T + g Z_\theta$$

The following additional notation (Holmboe, Forsythe, Gustin [3]) will be used in the discussions involving baric and anti-baric flow:

b_n	Normal pressure force
c_n	Coriolis force
$\dot{v}_n$	Centripetal acceleration
i	Path of flow in inertial flow

I. INTRODUCTION

The study of the effects of pressure variations on atmospheric wind systems occupies an important place in the meteorological literature. Articles which pursue this study in its many and varied forms appear regularly in meteorological journals. Textbooks in Dynamic Meteorology present the basic essentials of the problem and discuss the effects of changing pressure fields upon accelerations of the wind and the deviation of the wind from geostrophic or gradient flow.

S. Petterssen [5] pursued this study from the standpoint of the sensitivity of the wind field relative to pressure variations. His method of development consisted of solving the differentiated equations of motion by the introduction of geostrophic approximations of the first and second order and obtaining a measure of the sensitivity of the wind field relative to pressure variations. In his study it is shown that the sensitivity depends essentially upon the tangential and orthogonal curvature of the geostrophic streamlines, the lateral shear of the geostrophic wind, and upon terms analogous to the components of the isallobaric wind.

The use of geostrophic approximations in Petterssen's study suggests immediately that this particular development could possibly be pursued from the standpoint of the gradient wind, which is a more general case. This is the method that is attempted in this paper.

Since the gradient wind is the more general case, and the geostrophic wind merely a specialized form of the gradient wind, the mathematical development of this study follows very closely that of Petterssen. The notations introduced in Petterssen's paper were followed except where otherwise indicated, and the values given for the various terms in the equations of motion by Petterssen are used. The above were used intentionally by the author in order that the developments of the two studies would be similar and a comparison of the two cases could be made.

The departure from Petterssen's development comes at the point where the differentiated equations of motion are solved, and the sensitivity factor obtained. The gradient approximation is necessarily different from the geostrophic approximation due to the presence of centripetal acceleration in gradient flow. This inclusion of the centripetal acceleration introduces additional terms in the equations for the components of the wind and in the sensitivity factor. Furthermore, in addition to the types of pressure fields studied by Petterssen, the sensitivity factor in this study was applied to baric and anti-baric flow as defined in Holmboe, Forsythe, Gustin [3].

The results obtained are consistent with those of Petterssen. It was found that the sensitivity depends upon the factors shown by Petterssen (with the geostrophic shear replaced by the gradient shear) plus the curvature of trajectory of the air particle, the change of trajectory curvature normal to the path, and a factor of proportionality which arises from the

gradient approximations. It is further shown in this paper that this factor of proportionality, which relates the magnitudes of the centripetal acceleration and the pressure force in gradient flow, can assume great importance when it is assigned different values, for it can cause a reversal of the effects of the other factors upon the sensitivity of the gradient wind field.

II. THEORETICAL DEVELOPMENT OF SENSITIVITY FACTOR

With the x-axis toward the east and the y-axis toward the north and following the notation given in the table of symbols, the equations of frictionless motion (following Petterssen [5]) are:

$$\begin{aligned} \frac{du}{dt} &= -\frac{\partial \Psi}{\partial x} + \lambda v - \lambda' w \\ \frac{dv}{dt} &= -\frac{\partial \Psi}{\partial y} - \lambda u \end{aligned} \qquad (1)$$

or

$$\begin{aligned} \dot{u} - \lambda v + \lambda' w &= X \\ \dot{v} + \lambda u \qquad &= Y \end{aligned} \qquad (2)$$

Differentiating equations (2) with respect to time we have:

$$\begin{aligned} \ddot{u} - \lambda\dot{v} - v\dot{\lambda} + \lambda'\dot{w} + w\dot{\lambda}' &= \dot{X} \\ \ddot{v} + \lambda\dot{u} + u\dot{\lambda} \qquad &= \dot{Y} \end{aligned} \qquad (3)$$

$\dot{u}$ and $\dot{v}$ may be eliminated by substituting equations (2) into equations (3) thus obtaining

$$\begin{aligned} \ddot{u} + \lambda^2 u - \lambda Y - v\dot{\lambda} + \lambda'\dot{w} + w\dot{\lambda}' &= \dot{X} \\ \ddot{v} + \lambda^2 v - \lambda\lambda' w + \lambda X + u\dot{\lambda} \qquad &= \dot{Y} \end{aligned} \qquad (4)$$

We now have equations involving the time derivatives of the components of the horizontal pressure force. No further differentiation will be performed and we shall now consider the relative magnitudes of the terms appearing in equations (4).

Now $\lambda = 2\omega \sin\phi$ and $\dot{\lambda} = 2\omega\cos\phi \frac{d\phi}{dt} = \frac{\lambda' v}{E}$. Also $\lambda' = 2\omega\cos\phi$ and $\dot{\lambda}' = -2\omega\sin\phi\frac{d\phi}{dt} = -\frac{\lambda v}{E}$. In middle latitudes λ and λ' are about equal.

Representative values of the parameters of the various terms are

$\lambda \doteq \lambda'$ on order of magnitude of 10^{-4} sec^{-1}

u on the order of 10 meters/second,

v on the order of 10 meters/second,

w on the order of 10^{-1} meters/second, and

$\dot{w}$ on the order of 10^{-5} meters/second^2.

Using these values, the magnitudes of the terms in equations (4) are

$$\lambda^2 u = 10^{-7}\, m \cdot sec^{-3}$$

$$\lambda^2 v = 10^{-7}\, m \cdot sec^{-3}$$

$$v\dot{\lambda} = \frac{10^{-8}}{6.4}\, m \cdot sec^{-3}$$

$$u\dot{\lambda} = \frac{10^{-8}}{6.4}\, m \cdot sec^{-3}$$

$$\lambda' \dot{w} = 10^{-9}\, m \cdot sec^{-3}$$

$$|w\dot{\lambda}'| = \frac{10^{-10}}{6.4}\, m \cdot sec^{-3}$$

$$\lambda\lambda' w = 10^{-9}\, m \cdot sec^{-3}$$

It is seen that the magnitude of the terms $(-v\dot{\lambda} + \lambda'\dot{w} + w\dot{\lambda}')$ and $(-\lambda\lambda' w + u\dot{\lambda})$ are about one to two percent of the terms $\lambda^2 u$ and $\lambda^2 v$, and may be omitted. The differentiated equations of motion become

$$\begin{aligned} \ddot{u} + \lambda^2 u &= \dot{X} + \lambda Y \\ \ddot{v} + \lambda^2 v &= \dot{Y} - \lambda X \end{aligned} \qquad (5)$$

We may now rotate the axes such that the positive y-axis coincides with the horizontal pressure force. The only terms affected by such rotation are those involving λ' which have been shown to be negligible.

With this choice of coordinate axes, we may now represent (following Montgomery [4]) the gradient wind on an isentropic surface in the vector form

$$\mathbb{V}_{gr} = -\frac{1}{\lambda}\nabla\psi \times \mathbb{k} + \frac{1}{\lambda}\mathbb{k} \times \left(-v_{gr}^2 K_T \mathbb{m}\right) \tag{6}$$

with the scalar components

$$\begin{aligned} u_{gr} &= -\frac{1}{\lambda}\frac{\partial \psi}{\partial y} - u_{gr}^2 K_T \\ v_{gr} &= \frac{1}{\lambda}\frac{\partial \psi}{\partial x} \end{aligned} \tag{7}$$

With this same choice of axes, the components of the horizontal pressure force may be expressed as

$$\begin{aligned} X &= -\frac{\partial \psi}{\partial x} = -\lambda v_{gr} = 0 \\ Y &= -\frac{\partial \psi}{\partial y} = \lambda u_{gr} + u_{gr}^2 K_T = \lambda V_{gr} + V_{gr}^2 K_T \end{aligned} \tag{8}$$

If the motion is adiabatic but not necessarily horizontal; i.e., the motion is along the isentropic surface, we have

$$\begin{aligned} \dot{X} &= -\frac{\partial^2 \psi}{\partial x \partial t} - \frac{\partial^2 \psi}{\partial x^2} u - \frac{\partial^2 \psi}{\partial x \partial y} v \\ \dot{Y} &= -\frac{\partial^2 \psi}{\partial y \partial t} - \frac{\partial^2 \psi}{\partial x \partial y} u - \frac{\partial^2 \psi}{\partial y^2} v \end{aligned} \tag{9}$$

Now the gradient of the stream function following our notation is

$$\nabla \psi = \frac{\partial \psi}{\partial x}\mathbb{t} + \frac{\partial \psi}{\partial y}\mathbb{m} \tag{10}$$

Since $\nabla\psi$ coincides with the y-axis, $\boldsymbol{t}\frac{\partial}{\partial x}\cdot(\nabla\psi) = 0$.

$$\boldsymbol{t}\frac{\partial}{\partial x}\cdot\nabla\psi = \boldsymbol{t}\frac{\partial}{\partial x}\cdot\left(\frac{\partial\psi}{\partial x}\boldsymbol{t} + \frac{\partial\psi}{\partial y}\boldsymbol{m}\right) =$$

$$\boldsymbol{t}\cdot\left(\frac{\partial^2\psi}{\partial x^2}\boldsymbol{t} + \frac{\partial\psi}{\partial x}\frac{\partial\boldsymbol{t}}{\partial x} + \frac{\partial^2\psi}{\partial x\partial y}\boldsymbol{m} + \frac{\partial\psi}{\partial y}\frac{\partial\boldsymbol{m}}{\partial x}\right) =$$

$$\boldsymbol{t}\cdot\boldsymbol{t}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial y}K_i\right) + \boldsymbol{t}\cdot\boldsymbol{m}\left(\frac{\partial\psi}{\partial x}K_i + \frac{\partial^2\psi}{\partial x\partial y}\right) \tag{11}$$

The second term above is zero since $\boldsymbol{t}\cdot\boldsymbol{m} = 0$.

$$\therefore\ \boldsymbol{t}\frac{\partial}{\partial x}\cdot\nabla\psi = \boldsymbol{t}\cdot\boldsymbol{t}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial y}K_i\right) = 0$$

Since $\boldsymbol{t}\cdot\boldsymbol{t} \neq 0$, $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial y}K_i = 0$ and

$$\frac{\partial^2\psi}{\partial x^2} = -K_i\frac{\partial\psi}{\partial y} \tag{12}$$

Following a similar development using the relation

$$\boldsymbol{m}\frac{\partial}{\partial y}\cdot(\boldsymbol{k}\times\nabla\psi) = 0$$

we obtain

$$\frac{\partial^2\psi}{\partial x\partial y} = K_n\frac{\partial\psi}{\partial y} \tag{13}$$

Now

$$\frac{\partial\psi}{\partial y} = -\lambda u_{gr} - u_{gr}^2 K_T \tag{14}$$

and

$$\frac{\partial^2\psi}{\partial y^2} = -\lambda\frac{\partial u_{gr}}{\partial y} - 2u_{gr}K_T\frac{\partial u_{gr}}{\partial y} - u_{gr}^2\frac{\partial K_T}{\partial y}$$

or

$$\frac{\partial^2\psi}{\partial y^2} = -\lambda\frac{\partial V_{gr}}{\partial y} - 2V_{gr}K_T\frac{\partial V_{gr}}{\partial y} - V_{gr}^2\frac{\partial K_T}{\partial y} \tag{15}$$

At this point the additional notations

$$I_x = -\frac{1}{\lambda^2}\frac{\partial^2\psi}{\partial x\,\partial t} \quad \text{and} \quad I_y = -\frac{1}{\lambda^2}\frac{\partial^2\psi}{\partial y\,\partial t} \tag{16}$$

will be introduced. These are analogous to the components of the isallobaric wind as defined by Brunt and Douglas [1].

The equations (5) are well suited to the investigation of the sensitivity of the gradient wind field to pressure variations, for the terms therein involve the first time-derivatives of the pressure force which in turn involve the first and second space-derivatives of the pressure force. Formal solutions of these equations will not be sought, but assumptions will be made concerning the terms $\ddot{u}$ and $\ddot{v}$, whereby these terms may be eliminated and expressions for u and v obtained. These assumptions will take the form of first and second order gradient approximations. The reasoning set forth by Petterssen [5] in section 3 will be followed using, however, relationships that occur when the more general case of the gradient wind is being considered.

With the y-axis coinciding with the direction of the gradient of the stream function, the acceleration of the gradient wind is

$$\begin{aligned}\dot{\mathbb{V}} &= \dot{u}\,\mathbb{t} + \dot{v}\,\mathbb{m} \\ &= \dot{u}\,\mathbb{t} + u^2 K_T\,\mathbb{m}\end{aligned} \tag{17}$$

Since there is no tangential acceleration in true gradient flow, $\dot{u} = 0$ and

$$\dot{\mathbb{V}} = u^2 K_T\,\mathbb{m}$$

In gradient flow this is the centripetal acceleration, which is on the order of magnitude of the coriolis acceleration except near the poles and the equator; i.e.,

$$|\dot{\mathbb{V}}| = u^2 K_T \text{ is on the order of magnitude of } |2\Omega \times \mathbb{V}|. \tag{18}$$

Since we desire this relationship to hold not only at a certain instant, but at all times in this study, we may say

$$|\ddot{\mathbb{V}}| = \frac{d}{dt}\left(u^2 K_T\right) \text{ is on the order of magnitude of } \left|\frac{d}{dt}\left(2\Omega \times \mathbb{V}\right)\right|. \tag{19}$$

Equations (18) and (19) will be called gradient approximations of the first and second order respectively.

Since the acceleration can not be neglected in relation to other terms in the study of the gradient wind, expressions for it and its time derivative must be obtained and substituted in the appropriate places in equations (5). The gradient wind equation expresses the condition that the pressure gradient force, coriolis force, and centrifugal force balance (Haurwitz [2]). Using this condition we may introduce the additional gradient approximation

$$\dot{v} = u_{gr}^2 K_T = m\,Y \tag{20}$$

where m is a factor of proportionality between the pressure gradient force and the centrifugal force. Since it is desired that such a proportionality exist throughout all times for purposes of this investigation, we may say also

m const.

$$\ddot{v} = \frac{d}{dt}\left(u_{gr}^2 K_T\right) = m\,\dot{Y} \tag{21}$$

This is equivalent to saying that the centrifugal force balances a certain portion of the pressure force and will continue to balance a the same certain portion during changes with respect to time.

Substituting the gradient approximation in equations (5) we have

$$0 + \lambda^2 u = \dot{X} + \lambda Y$$
$$m\dot{Y} + \lambda^2 v = \dot{Y} - \lambda X$$

or

$$\lambda^2 u = \dot{X} + \lambda Y$$
$$\lambda^2 v = M\dot{Y} - \lambda X \qquad \text{where } M = 1 - m. \tag{22}$$

Substituting equations (9) and (10) into equations (22) and further substituting (12), (13), (15), and (16) we obtain

$$\lambda^2 u = \lambda^2 I_x - K_i\left(\lambda V_{gr} + V_{gr}^2 K_T\right)u - K_n\left(-\lambda V_{gr} - V_{gr}^2 K_T\right)v + \lambda\left(\lambda V_{gr} + V_{gr}^2 K_T\right)$$
$$\lambda^2 v = M\left[\lambda^2 I_y - K_n\left(-\lambda V_{gr} - V_{gr}^2 K_T\right)u - \left(-\lambda \frac{\partial V_{gr}}{\partial y} - 2\lambda V_{gr} K_T \frac{\partial V_{gr}}{\partial y} - V_{gr}^2 \frac{\partial K_T}{\partial y}\right)v\right], \tag{23}$$

or

$$0 = Au + Bv + E$$
$$0 = Cu + Dv + F. \tag{24}$$

Factoring λ^2 from each of the equations (24), the coefficients are

$$A = -1 - K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)$$
$$B = K_n\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)$$
$$C = M\left[K_n\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]$$
$$D = -1 + M\left(\frac{1}{\lambda}\frac{\partial V_{gr}}{\partial y} + \frac{2}{\lambda^2} V_{gr} K_T \frac{\partial V_{gr}}{\partial y} + \frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}\right) \tag{25}$$
$$E = I_x + V_{gr} + \frac{1}{\lambda} V_{gr}^2 K_T$$
$$F = M I_y$$

and it is seen that $C = MB$.

It is convenient at this point to give arbitrary, but representative, values to the factors appearing in the above coefficients, in order that we may determine the order of magnitude of these coefficients. This will be of assistance in the qualitative discussion of the various types of pressure fields.

Using the following values

$$K_i = \pm \frac{10^{-5}}{5} \; m^{-1} \qquad \frac{\partial V_{gr}}{\partial y} = \pm 10^{-4} \; sec^{-1}$$

$$K_T = \pm 10^{-6} \; m^{-1} \qquad \lambda = 10^{-4} \; sec^{-1}$$

$$V_{gr} = 10 \; m \cdot sec^{-1} \qquad M = 0.5$$

$$K_n = \pm 10^{-6} \; m^{-1} \qquad I_x = 2 \; m \cdot sec^{-1}$$

$$\frac{\partial K_T}{\partial y} = -10^{-12} \; m^{-2} \qquad I_y = 2 \; m \cdot sec^{-1}$$

we find	Cyclonic Flow	Anticyclonic Flow
$A =$	-1.22	-0.82
$B =$	0.11	0.09
	-0.11	-0.09
$C =$	0.06	0.04
	-0.06	-0.04
$D =$	-1.61	-1.41
	-0.41	-0.61
$E =$	$13 \; m \cdot sec^{-1}$	$11 \; m \cdot sec^{-1}$
$F =$	$1 \; m \cdot sec^{-1}$	$1 \; m \cdot sec^{-1}$

Solving equations (24) for u and v we have:

$$u = \frac{BF - DE}{AD - MB^2} \quad \text{and} \quad v = \frac{MBE - AF}{AD - MB^2} \tag{26}$$

Substituting coefficients and absorbing negative signs the equations become:

$$u = \frac{DE + BF}{AD - MB^2} \quad \text{and} \quad v = \frac{AF + MBE}{AD - MB^2} \tag{27}$$

and the coefficients are:

	Cyclonic Flow	Anticyclonic Flow
$A = 1 + K_i\left(\frac{1}{\lambda}V_{gr} + \frac{1}{\lambda^2}V_{gr}^2 K_T\right)$	1.22	0.82
$B = K_n\left(\frac{1}{\lambda}V_{gr} + \frac{1}{\lambda^2}V_{gr}^2 K_T\right)$	-0.11 0.11	-0.09 0.09
$C = MB$	-0.06 0.06	-0.04 0.04
$D = 1 - M\left[\left(\frac{1}{\lambda}\frac{\partial V_{gr}}{\partial y} + \frac{2}{\lambda^2}V_{gr}K_T\frac{\partial V_{gr}}{\partial y}\right) + \frac{1}{\lambda^2}V_{gr}^2\frac{\partial K_T}{\partial y}\right]$	1.61 0.41	1.41 0.61
$E = I_x + V_{gr} + \frac{1}{\lambda}V_{gr}^2 K_T$	13 $m \cdot sec^{-1}$	11 $m \cdot sec^{-1}$
$F = M I_y$	1 $m \cdot sec^{-1}$	1 $m \cdot sec^{-1}$

which are the forms of the equations and the coefficients therein which will be used in the following discussion.

Since we have chosen the x-axis tangent to gradient flow at all times, v of equations (27) is the cross-streamline component of the actual wind. It is apparent that this component increases or decreases as the denominator decreases or increases respectively, hence the quantity

$$S_{gr} = \frac{1}{AD - MB^2} \tag{28}$$

may be considered as a measure of the sensitivity of the gradient wind relative to pressure variations. It will be shown in the next chapter how the tangential and normal components of flow as defined by equations (27) react to the sensitivity factor in various types of pressure fields.

III. TYPES OF PRESSURE FIELDS

From the equations (27) it is seen that the deviation of the wind from the gradient wind depends partly upon each of the following factors:

(a) terms analogous to the isallobaric components I_x and I_y ;

(b) the lateral shear of the gradient wind, $\frac{\partial V_{gr}}{\partial y}$;

(c) the tangential curvature of the streamlines, K_i ;

(d) the orthogonal curvature of the streamlines, K_n ;

(e) the curvature of the trajectory of the air particle, K_T ;

(f) the change of the trajectory curvature normal to the flow, $\frac{\partial K_T}{\partial y}$; and

(g) the factor $M = 1 - m$.

The effect of each of these factors will be shown qualitatively for various types of pressure fields, and the sensitivities for the various cases will be compared. It will be convenient for this discussion to begin with the simpler, more idealized types of systems and work toward the more complex systems. Following these discussions, equations (27) will be applied to baric and anti-baric flow as defined by Holmboe, Forsythe, Gustin [3], and the results discussed qualitatively.

1. Curved concentric streamlines with no gradient shear.

In this case $K_n = \frac{\partial V_{gr}}{\partial y} = 0$ and equations (27) reduce to

$$u = \frac{\left[\left(I_x + V_{gr} + \frac{1}{\lambda} V_{gr}^2 K_T\right)\right]\left[1 - M\left(\frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}\right)\right]}{\left[1 + K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]\left[1 - M\left(\frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}\right)\right]} \tag{29}$$

$$v = \frac{M I_y \left[1 + K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]}{\left[1 + K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]\left[1 - M\left(\frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}\right)\right]}$$

Using the values of the coefficients as determined in Chapter I, we have $u = 10.7$ meters/second and $v = 1.0$ meter/second for cyclonic flow and $u = 13.4$ meters/second and $v = 1.0$ meter/second for anticyclonic flow. It is seen that the motion is more sensitive to pressure variations when the curvature of the streamlines (and trajectory) is anticyclonic than when these curvatures are cyclonic. The cross-streamline component v is seen to depend also upon the factors M, I_y, and $\frac{\partial K_T}{\partial y}$. The fact that the term A can be cancelled from the equation for v leads to the result that the cross-streamline component would be the same for both the cyclonic and anticyclonic cases.

2. Curved concentric streamlines with gradient shear.

With $K_n = 0$ equations (27) become

$$u = \frac{\left[I_x + V_{gr} + \frac{1}{\lambda} V_{gr}^2 K_T\right]\left[1 - M\left(\frac{1}{\lambda}\frac{\partial V_{gr}}{\partial y} + \frac{2}{\lambda^2} V_{gr} K_T \frac{\partial V_{gr}}{\partial y} + \frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}\right)\right]}{\left[1 + K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]\left[1 - M\left(\frac{1}{\lambda}\frac{\partial V_{gr}}{\partial y} + \frac{2}{\lambda^2} V_{gr} K_T \frac{\partial V_{gr}}{\partial y} + \frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}\right)\right]}$$

$$v = \frac{M I_y \left[1 + K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]}{\left[1 + K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]\left[1 - M\left(\frac{1}{\lambda}\frac{\partial V_{gr}}{\partial y} + \frac{2}{\lambda^2} V_{gr} K_T \frac{\partial V_{gr}}{\partial y} + \frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}\right)\right]} \tag{30}$$

the orders of magnitude of which are (for cyclonic motion): $u = 10.7 \cdot m \cdot sec^{-1}$ and $v = 2.44\ m \cdot sec^{-1}$ with anticyclonic shear, and $u = 10.7\ m \cdot sec^{-1}$ and $v = 0.62\ m \cdot sec^{-1}$ with cyclonic shear. It is shown by equations (30) that the motion is more sensitive in the case of anticyclonic shear than in the case of cyclonic shear, due to the decrease in the value of the denominator when $\frac{\partial V_{gr}}{\partial y} > 0$. Since the term D can be cancelled from the equation for u, in both cases 1 and 2, it is seen that the gradient shear term does not affect this component of equations (29) and (30).

Combining the results of the two cases that have been discussed, it is seen that the gradient wind is more sensitive to pressure variations when the curvature of the streamlines and trajectory and the gradient shear are anticyclonic.

3. Converging or diverging streamlines.

This case is represented by equations (27)

$$u = \frac{DE + BF}{AD - MB^2} \qquad \text{and} \qquad v = \frac{AF + MBE}{AD - MB^2}$$

where none of the terms vanish and all factors exert their influences upon the sensitivity and the motion of the wind. The effects of the trajectory curvature, streamline curvature, and gradient shear have already been discussed. The orthogonal curvature of the streamlines, which makes its appearance in this type of pressure field, serves always to decrease the value of the denominator and therefore to increase the sensitivity, since the only term in which it appears in the denominator is squared.

In the above cases the factor $\frac{\partial K_T}{\partial y}$, which appears only in the term D, acts always to decrease the sensitivity in the case of concentric trajectories or where the radii of curvature of the trajectories decrease as we proceed toward the centers of the systems normal to the flow. This is due to the fact that its sign remains negative for both cyclonic and anticyclonic trajectories. In certain cases of converging and diverging trajectories, however, the radii of curvature will increase at certain points as we proceed toward the centers of the systems. In such cases the effect of this factor on the sensitivity would be reversed. Considering

the magnitude of the term, $\frac{1}{\lambda^2} V_{gr}^2 \frac{\partial K_T}{\partial y}$, as compared to the other terms of D, however, it is seen that in such cases the effect of the sign change in $\frac{\partial K_T}{\partial y}$ would introduce only very small and, in the majority of cases, negligible contributions to the overall effect of term D.

the magnitude of the term, $\frac{1}{K_1} V_{gm}^{2} \frac{2K^{*}}{\bar{q}^{2} q}$, is compared to the other terms of D, however, it is seen that if such were the effect of the sign change in $\frac{2K^{*}}{\bar{q}^{2} q}$ would introduce only very small and, in the majority of cases, negligible contributions to the overall effect of term D.

IV. THE M-FACTOR IN BARIC AND ANTIBARIC FLOW

Little has been said up to now of the factor M. In the foregoing cases the value of M assigned in Chapter I has been used. This value of M corresponds to the situation where the centripetal acceleration balances a fractional part of the pressure gradient force. This value of M reduces the absolute value of the terms in which it appears and thus modifies the extent to which those terms affect the sensitivity and the motion of the wind. The effect of M when it takes on values outside the range $0 < M < 1$ is conveniently shown in the cases of baric and anti-baric flow which will now be discussed.

Following Holmboe, Forsythe, and Gustin [3], baric flow exists where the normal pressure force is opposite the horizontal coriolis force and anti-baric flow occurs where the normal pressure force and the horizontal coriolis force are in the same direction. Where the horizontal centripetal acceleration is opposite to the horizontal coriolis force, the flow is cyclonic; and where the horizontal centripetal acceleration is along the horizontal coriolis force, the flow is anticyclonic. It follows from his definitions that cyclonic and geostrophic flow are always baric, whereas anticyclonic flow may be baric or anti-baric.

In presenting these cases the following restrictions and assumptions are made, again following Holmboe, Forsythe, and Gustin [3]:

(a) the y-axis coincides with the direction of the gradient of the stream function;

(b) the latitude is fixed, thereby making the coriolis force constant;

(c) the wind speed is compatible with the pressure gradient force; i.e., no super- or sub-gradient flow occurs;

(d) the flow is along the x-axis; and

(e) the acceleration is purely centripetal.

The definition of gradient flow involves a balance of the three forces: pressure, coriolis, and centrifugal. Since we have fixed the value of the coriolis force by (b) above and the value of the pressure gradient force is assigned in each of the following cases, the value of the centripetal acceleration is determined by the relationship

$$\dot{v}_n = b_n + c_n \tag{31}$$

for each case considered.

1. Baric cyclonic flow.

This is shown in Figure 1. In this case b_n is along $-c_n$. $|b_n| > |c_n|$ and in the gradient approximation

$$\dot{v}_n = m b_n \tag{20}$$

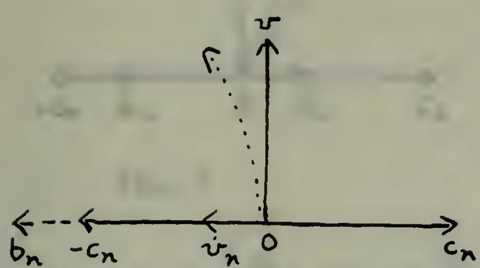

Fig. 1

m lies in the range $0 < m < 1$ and M lies in the same range of values. The effect of the factor M in baric cyclonic flow is merely to reduce the magnitude of the terms in which it appears. The other factors (streamline curvature, trajectory curvature, orthogonal curvature, change in trajectory curvature, gradient shear, and I_x and I_y) affect the sensitivity in the manner that has been heretofore described.

2. Baric geostrophic flow.

As shown in Figure 2, b_n is at $-c_n$ and $\dot{v}_n$ is zero. In the gradient approximation

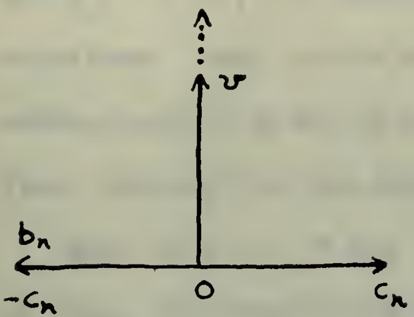

Fig. 2

$$\dot{v}_n = m\, b_n \tag{20}$$

$m = 0$ since b_n is not zero. Therefore $M = 1$. In this case the equations (27) reduce to Petterssen's equations (14), which were used in his presentation of the sensitivity of the geostrophic wind field relative to pressure variations.

3. Baric anticyclonic flow.

Figure 3 shows the relationships of the three forces for this case.

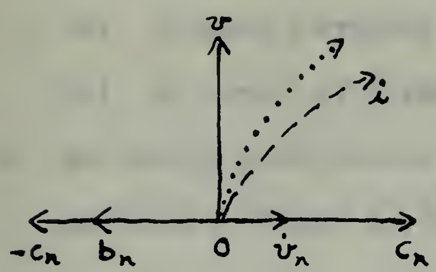

Fig. 3

b_n is along $-c_n$ and is restricted to values lying between $-c_n$ and 0. Therefore $\dot{v}_n$ must necessarily lie in the range $0 < \dot{v}_n < c_n$. The relationship between b_n and $\dot{v}_n$ is $\dot{v}_n \lesseqgtr b_n$. In the gradient approximation

$$\dot{v}_n = m\, b_n \tag{20}$$

we see that $m \lesseqgtr 1$ and $M \gtreqless 0$.

The case of $m < 1$ and $M < 1$ has been discussed above in detail and will not be dealt with further at this point.

In the case of $m = 1$ and $M = 0$, equations (27) reduce to

$$u = \frac{I_x + V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T}{1 + K_i\left(\frac{1}{\lambda} V_{gr} + \frac{1}{\lambda^2} V_{gr}^2 K_T\right)} \tag{32}$$

$$v = 0$$

The component v is seen to vanish since the numerator is multiplied in its entirety by M. This is analogous to the case discussed by Petterssen [5, p. 20], in which he defines "balanced motion" by means of his equations. Thus, in the meaning of Petterssen, this sub-case of baric anticyclonic flow may be considered the "balanced motion" case for gradient flow, arising from the restriction imposed by the gradient approximation.

When $m > 1$, $M < 0$, and the terms multiplied by M undergo a reversal of sign. The sensitivity effects of those terms and the factors appearing therein are thereby necessarily reversed. Thus we find less sensitivity is associated with:

(a) anticyclonic curvature and shear ($K_i < 0, K_T < 0, \frac{\partial V_{gr}}{\partial y} > 0$) and

(b) orthogonal curvature ($K_n \gtrless 0$);

and greater sensitivity associated with:

(c) cyclonic curvature and shear ($K_i > 0, K_T > 0 \frac{\partial V_{gr}}{\partial y} < 0$)

(d) the change of trajectory curvature normal to the flow ($\frac{\partial K_T}{\partial y}$).

4. Anti-baric anticyclonic flow.

As shown in Figure 4, b_n is to the right of zero and hence lies along c_n. Then $\dot{v}_n > b_n$ and $\dot{v}_n = m b_n$ where $m>1$ and $M<0$. This is the same as was discussed under baric anticyclonic flow where the effect of the factor M was to reverse the effect of the various factors which appear in the terms multiplied by M.

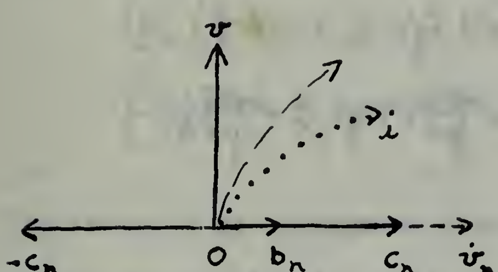

Fig. 4

In this case, however, the reversal effect is more pronounced due to the greater absolute value of M, and this reversal effect continues to become more pronounced as M increases in absolute value.

5. Inertial flow.

The case of inertial flow is shown in Figure 5. In this case, the horizontal pressure force is at zero and the centripetal acceleration and the coriolis forces remain as the two balancing forces. Applying the gradient approximation

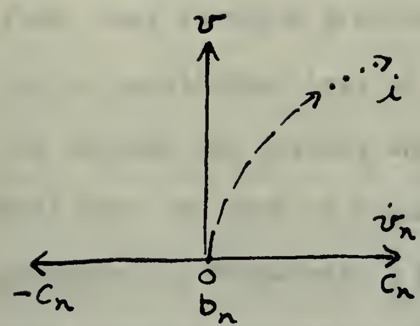

Fig. 5

$$\dot{v}_n = m\, b_n \tag{20}$$

we find that the factor m and hence the factor M become infinity. Since M appears in both the numerator and denominator of each of the components u and v of equations (27), we may divide the numerators and denominators of the two equations by M. Then, taking the limit we have

$$\begin{aligned}
u = {} & I_y K_n\left(\tfrac{1}{\lambda} V_{gr} + \tfrac{1}{\lambda^2} V_{gr}^2 K_T\right) - \left(I_x + V_{gr} + \tfrac{1}{\lambda} V_{gr}^2 K_T\right) \\
& \left(\tfrac{1}{\lambda}\tfrac{\partial V_{gr}}{\partial y} + \tfrac{2}{\lambda^2} V_{gr} K_T \tfrac{\partial V_{gr}}{\partial y} + \tfrac{1}{\lambda^2} V_{gr}^2 \tfrac{\partial K_T}{\partial y}\right) \Big/ \left[1 + K_i\left(\tfrac{1}{\lambda} V_{gr} + \tfrac{1}{\lambda^2} V_{gr}^2 K_T\right)\right] \\
& \left[-\left(\tfrac{1}{\lambda}\tfrac{\partial V_{gr}}{\partial y} + \tfrac{2}{\lambda^2} V_{gr} K_T \tfrac{\partial V_{gr}}{\partial y} + \tfrac{1}{\lambda^2} V_{gr}^2 \tfrac{\partial K_T}{\partial y}\right)\right] - \left[K_n\left(\tfrac{1}{\lambda} V_{gr} + \tfrac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]^2 \\
v = {} & I_y\left[1 + K_i\left(\tfrac{1}{\lambda} V_{gr} + \tfrac{1}{\lambda^2} V_{gr}^2 K_T\right)\right] + \left[I_x + V_{gr} + \tfrac{1}{\lambda} V_{gr}^2 K_T\right] \\
& \left[K_n\left(\tfrac{1}{\lambda} V_{gr} + \tfrac{1}{\lambda^2} V_{gr}^2 K_T\right)\right] \Big/ \left[1 + K_i\left(\tfrac{1}{\lambda} V_{gr} + \tfrac{1}{\lambda^2} V_{gr}^2 K_T\right)\right] \\
& \left[-\left(\tfrac{1}{\lambda}\tfrac{\partial V_{gr}}{\partial y} + \tfrac{2}{\lambda^2} V_{gr} K_T \tfrac{\partial V_{gr}}{\partial y} + \tfrac{1}{\lambda^2} V_{gr}^2 \tfrac{\partial K_T}{\partial y}\right)\right] - \left[K_n\left(\tfrac{1}{\lambda} V_{gr} + \tfrac{1}{\lambda^2} V_{gr}^2 K_T\right)\right]^2
\end{aligned} \tag{33}$$

The factor M is eliminated from these equations. The other factors that affect the sensitivity still remain and have the effects that have already been shown.

V. CONCLUSIONS

The results of this study have been set forth for each case considered. The types of pressure fields discussed were, with the exception of inertial flow, ones in which gradient flow occurs. The case of geostrophic flow is only a specialized type of gradient flow. This chapter, therefore, will not include the results that were obtained and discussed in Chapter II, but will take the form of a qualitative comparison of the geostrophic case as presented by Petterssen [5] and the gradient case undertaken in this study, with comments concerning a more comprehensive analysis of this study and other methods of development of the theory which occurred to, but were not undertaken by the author in this present paper.

As stated in the introduction, the development of the theory followed closely the development of Petterssen's theory in the geostrophic case. We have seen, however, that differences arose in the type of approximation that was formulated and the types of cases that were considered. A qualitative comparison of the two cases, therefore, is concerned mainly with the effects of the additional factors that appear in the case of the gradient wind; viz., the centripetal acceleration, the gradient shear, the change in trajectory curvature normal to the flow, and the factor M.

The sensitivity factor, which is the common denominator of the two basic equations (27), contains only the terms A, D, M, and B. The terms A and B contain the centripetal acceleration term, which does not appear in the geostrophic case. The term D contains the gradient shear term, which, in itself, involves two more terms than the geostrophic shear, due to the

presence of the centripetal acceleration. The term D, furthermore, contains the factor M. A qualitative comparison of the two cases readily shows that, in the large scale motions normally encountered in the atmosphere, where the curvature of the trajectory of the air particle is of the same sign as the streamline curvature, the effect of the additional terms (with the exception of M and $\frac{\partial K_T}{\partial y}$) is to produce a greater degree of magnitude of the sensitivity of the wind field in gradient flow than in geostrophic flow with the same degree of pressure variation. The effects of these terms would become even more pronounced with an increase in wind velocity, since the centripetal acceleration involves the square of the wind velocity, and furthermore the factor M is reduced as the centripetal acceleration increases. The factor M generally appears in the range $0<M<1$ and therefore tends to reduce the enhanced sensitivity brought about by the other two factors. The factor $\frac{\partial K_T}{\partial y}$, as has been shown, acts always to decrease the sensitivity in all systems, cyclonic and anticyclonic, where the curvature of the trajectories increases as one proceeds toward the center of curvature. This is the case in all systems of concentric trajectories and in some systems of diverging and converging trajectories. In other cases of diverging and converging trajectories, the curvature may decrease normal to the flow and toward the center of curvatures. In these cases, due to the change of sign of $\frac{\partial K_T}{\partial y}$, it was seen that the effect of that factor on sensitivity is reversed. It was also shown, however, that the magnitude of the term involving $\frac{\partial K_T}{\partial y}$ is only a fractional part of the remainder of the terms of the coefficient D, and its overall effect would be small in all large-scale currents, and

would only become appreciable near the centers of small closed systems, possibly being a contributing factor to the erratic behavior of winds near system centers.

The actual degree to which the sensitivity is magnified in the gradient case over the geostrophic case would, of course, involve lengthy numerical computations using values of the factors in systems appearing on the weather maps, and such computations were not attempted in this study. It would seem that the most interesting comparison of the two would be in regions of slight to moderate curvature of the streamlines where the geostrophic wind is often used as an approximation to the true wind.

In this study the gradient approximations of the first and second order were established as relationships involving the centripetal acceleration and the coriolis force. This relationship was used as a basis for the establishment of a proportionality between the centripetal acceleration and the pressure force, and in this form it was substituted into the differentiated equations of motion. Since the gradient wind involves a balance among the three forces -- pressure, centrifugal, and coriolis -- it follows that there should be a means of equating the centripetal acceleration and the coriolis force by means of a factor of proportionality and substituting in the differentiated equations of motion. This was attempted but not carried to completion in this study as the method of development used in this paper was preferred.

Petterssen stated in his conclusions that the synoptic usefulness of the sensitivity factor remained to be determined, but that work along those lines was at that time nearing completion. Should the geostrophic sensitivity factor be established as a useful synoptic tool, additional computations and comparisons as suggested above would establish whether or not the gradient sensitivity factor would be a more useful tool in those regions where the geostrophic wind is not a good approximation to the true wind.

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